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WELLPOSEDNESS OF ABSTRACT CAUCHY PROBLEMS FOR SECOND ORDER DIFFERENTIAL EQUATIONS

BY

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ABSTRACT

This paper concerns the abstract Cauchy problem (ACP) for an evolution equation of second order in time. Let A be a closed linear operator with domain $D(A)$ dense in a Banach space X. We first characterize the exponential wellposedness of ACP on $D(A^{k+1})$, $k \in \mathrm{N}$. Next let $\{C(t);\ t \in \mathbf{R}\}$ be a family of generalized solution operators, on $[D(A^k)]$ to X, associated with an exponentially wellposed ACP on $D(A^{k+1})$. Then we define a new family $\{T(t);\ \mathrm{Re}\, t > 0\}$ by the abstract Weierstrass formula. We show that $\{T(t)\}$ forms a holomorphic semigroup of class (H_k) on X.

1. Introduction

Let X be a complex Banach space with norm $\|\cdot\|$. Let A be a closed linear operator with domain $D(A)$ dense in X. In this paper we shall restrict ourselves to the case in which A has a nonempty resolvent set $\rho(A)$. Let Y be a linear manifold in $D(A)$ and consider the differential equation in X

$$(1.1)\qquad (d^2/dt^2)u(t) = Au(t), \qquad t \in \mathbf{R}.$$

By an *abstract Cauchy problem on Y* we mean the following:

ACP. Given two elements $x, y \in Y$, find an X-valued function $u(t) = u(t; x, y)$ defined on $\mathbf{R}$ such that

(A_1) $u(t)$ is twice continuously differentiable in t,

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(A_2) $u(t) \in D(A)$, $u(t)$ satisfies (1.1),

(A_3) $u(0) = x$, $(d/dt)u(0) = y$.

A function $u(t)$ satisfying (A_1)–(A_3) is called a *solution* to ACP.

We say that ACP on Y is *exponentially wellposed* if for every $x, y \in Y$ there is a unique solution $u(t) = u(t; x, y)$ to ACP on Y, satisfying

(EG) There is a constant $\omega > 0$, called the constant of exponential growth, such that

$$\| u(t) \| + \| u''(t) \| = O(e^{\omega |t|}) \qquad \text{as } |t| \to \infty,$$

where $u''(t)$ stands for $(d^2/dt^2)u(t)$.

The main purpose of this paper is to characterize the exponential wellposedness of ACP on $D(A^{k+1})$, $k \in \mathbf{N}$. Namely, we shall present a criterion for ACP on $D(A^{k+1})$ to be exponentially wellposed. Here it should be noted that ACP on $D(A)$ is exponentially wellposed if and only if A is the generator of a cosine family $\{C(t); t \in \mathbf{R}\}$; in this case a unique solution to ACP is given by

$$u(t; x, y) = C(t)x + \int_0^t C(s)y\,ds$$

(see Fattorini [4]).

Before stating our result, let us briefly sketch the same problem for first order differential equations of the form

$$(d/dt)u(t) = Au(t), \qquad u(0) = x. \tag{1.2}$$

As a classical result of Hille [6] and Phillips [13] it is well known that ACP on $D(A)$ is (exponentially) wellposed if and only if A is the generator of a (C_0)-semigroup $\{T(t); t \geqq 0\}$; in this case a unique solution to ACP is given by $u(t; x) = T(t)x$ (see e.g. Goldstein [5], Krein [8] or Pazy [12]). The exponential wellposedness of ACP on $D(A^{k+1})$, $k \in \mathbf{N}$, was treated by Oharu [10] and Sanekata [14]: ACP on $D(A^{k+1})$ for (1.2) is exponentially wellposed if and only if

(F_1) there is $\omega > 0$ such that $\rho(A) \supset \{\lambda \in \mathbf{C}; \operatorname{Re}\lambda > \omega\}$,

(F_2) there is a constant $M > 0$ such that for $x \in D(A^k)$,

$$\| R(\lambda; A)^n x \| \leqq M(\operatorname{Re}\lambda - \omega)^{-n} \| x \|_k, \qquad \operatorname{Re}\lambda > \omega, \quad n \in \mathbf{N},$$

where $R(\lambda; A)$ denotes the resolvent of A and

$$\| x \|_k = \sum_{j=0}^{k} \| A^j x \|, \qquad A^0 x = x.$$

The "if" part of the characterization was previously proved by Oharu [10] and the "only if" part was shown later in Sanekata [14]. Recently, these facts have been extensively studied by Neubrander [9] through the notion of integrated semigroups.

The first result in this paper is a second-order analogue of the Oharu–Sanekata theorem and is stated as follows: ACP on $D(A^{k+1})$ for (1.1) is exponentially wellposed if and only if

(I) there is $\omega > 0$ such that $\rho(A) \supset \{\lambda^2 \in \mathbf{C};\ \mathrm{Re}\,\lambda > \omega\}$,

(II) there is a constant $M > 0$ such that for $x \in D(A^k)$,

$$\| (d/d\lambda)^{n-1}[\lambda R(\lambda^2; A)x] \| \leqq M(n-1)!\,(\mathrm{Re}\,\lambda - \omega)^{-n} \| x \|_k, \qquad \mathrm{Re}\,\lambda > \omega, \quad n \in \mathbf{N}.$$

The "if" part of the above statement was proved in [16]. Thus, this paper concerns the "only if" part.

In [16] we have constructed a family $\{C(t);\ t \in \mathbf{R}\}$ of solution operators to prove that ACP on $D(A^{k+1})$ is exponentially wellposed. $C(t)$ is a bounded linear operator on $[D(A^k)]$ to X:

$$\| C(t)x \| \leqq Me^{\omega t} \| x \|_k, \qquad x \in D(A^k), \quad t \in \mathbf{R}.$$

Thus, we can define a new family $\{T(t);\ \mathrm{Re}\,t > 0\}$ of linear operators by the abstract Weierstrass formula

$$T(t)x = \frac{1}{\sqrt{\pi t}} \int_0^\infty e^{-s^2/4t} C(s)x\,ds. \tag{1.3}$$

The second purpose in this paper is to make the properties of $\{T(t)\}$ clear. Let $b \in \rho(A)$. Then we shall rewrite (1.3) as

$$(\pi t)^{1/2} T(t)x = \sum_{j=0}^{k} \binom{k}{j} b^{k-j} \left(\frac{-1}{2t} \right)^j \int_0^\infty H_{2j}(s/\sqrt{2t}) \exp\left(-\frac{s^2}{4t} \right) C(s) R(b; A)^k x\,ds,$$

where $H_{2j}(r)$ is the Hermite polynomial of degree $2j$. Using this formula, we can show that $\{T(t);\ \mathrm{Re}\,t > 0\}$ is a holomorphic semigroup of bounded linear operators on X. Furthermore, we prove that for any $\varepsilon > 0$, $\{T(t);\ |\arg t| < (\pi/2) - \varepsilon\}$ forms a semigroup of class (H_k) and A is its complete infinitesimal generator. Roughly speaking, $\{T(t);\ |\arg t| < \alpha\}$ is of class (H_k) if for any $\delta > 0$,

$$\| T(t) \| = O(|t|^{-k}) \quad \text{and} \quad \| T(t)x - x \| \to 0 \quad \text{for } x \in D(A^k)$$

as $|t| \to 0$ with $|\arg t| \leqq \alpha - \delta$. This result is a generalization of the following: If A is the generator of a cosine family on X then A is also the generator of a holomorphic semigroup of class (C_0) on X (cf. Goldstein [5], II–§8; see also Yosida [18]).

This paper consists of four sections and appendix. In Section 3 we characterize the exponential wellposedness of ACP on $D(A^{k+1})$, $k \in \mathbf{N}$. As a preliminary we discuss in Section 2 solution operators of ACP. Section 4 is devoted to abstract Weierstrass semigroups associated with families of solution operators of ACP. Finally, we show in the Appendix that the original definition of a semigroup of class (H_n) is equivalent to a slightly simplified one.

2. Preliminaries

First we state some consequences of the exponential wellposedness. In the definition of exponential wellposedness in Section 1, we have assumed the exponential growth of solutions and their second derivatives as condition (EG). But, as is easily seen, (EG) is equivalent to

$$\| u(t) \| + \| u'(t) \| + \| u''(t) \| = O(e^{\omega |t|}) \qquad \text{as } |t| \to \infty. \tag{2.1}$$

Now let A be a closed linear operator with domain $D(A)$ and range $R(A)$ in X. Suppose that ACP on $D(A^{k+1})$, $k \in \mathbf{N}$, is exponentially wellposed and let $u(t; x, y)$ be a unique solution to ACP with $u(0) = x$ and $u'(0) = y$. Then it follows from the linearity and uniqueness that

$$u(t; x, y) = u(t; x, 0) + u(t; 0, y).$$

Moreover, the equivalence of (EG) to (2.1) implies that

$$u(t; 0, y) = \int_0^t u(s; y, 0)ds.$$

Thus, the solution to ACP with $u(0) = 0$ and $u'(0) = y$ can be represented as the indefinite integral of the solution to ACP with $u(0) = y$ and $u'(0) = 0$. So, it suffices to consider the ACP with $u'(0) = 0$, and we come to a definition of solution operators. Let $U(t)$ be a linear operator assigning the solution $u(t; x, 0)$ to an initial value $u(0) = x \in D(A^{k+1})$:

$$U(t)x = u(t; x, 0), \qquad t \in \mathbf{R}. \tag{2.2}$$

We shall clarify some properties of the family $\{U(t); t \in \mathbf{R}\}$.

LEMMA 2.1. *Let A be a closed linear operator in X. Assume that ACP on $D(A^{k+1})$ is exponentially wellposed and let $\{U(t)\}$ be the associated family of solution operators. Let $j \in \mathbf{N}$. Then for $x \in D(A^{k+j+1})$, $U(t)x$ is $2j$-times continuously differentiable, with values in $D(A^j)$, and*

$$(d^2/dt^2)^j U(t)x = A^j U(t)x = U(t)A^j x, \qquad t \in \mathbf{R}. \tag{2.3}$$

PROOF. We can prove the assertion by induction. Let $j = 1$. Then the first equality in (2.3) is obvious. We have to show that

$$AU(t)x = U(t)Ax \qquad \text{for } x \in D(A^{k+2}). \tag{2.4}$$

Since $Ax \in D(A^{k+1})$, we have

$$(d^2/dt^2)U(t)Ax = AU(t)Ax.$$

It follows from the closedness of A that

$$U(t)Ax = Ax + A\int_0^t (t-s)U(s)Axds = Az(t), \tag{2.5}$$

where we have set

$$z(t) = x + \int_0^t (t-s)U(t)Axds.$$

Noting (2.5), it is easy to see that

$$(d^2/dt^2)z(t) = U(t)Ax = Az(t),$$

$$z(0) = x, \qquad z'(0) = 0,$$

and

$$\| z(t) \| + \| z''(t) \| = O(e^{\omega|t|}) \qquad \text{as } |t| \to \infty.$$

Thus, by the uniqueness of solutions, we obtain $z(t) = U(t)x$ and $U(t)Ax = Az(t) = AU(t)x$. The rest of the proof is easy. Q.E.D.

Let A be a closed linear operator in X. Then for $k \in \mathbf{N}$ we may regard $D(A^k)$ as a Banach space with respect to the norm

$$\| x \|_k = \| x \| + \| Ax \| + \cdots + \| A^k x \| .$$

We write $[D(A^k)]$ for this Banach space.

LEMMA 2.2. *Let A and $\{U(t)\}$ be as in Lemma 2.1. Assume further that A has a nonempty resolvent set $\rho(A)$. Then there is a constant $M > 0$ such that for all $x \in D(A^{k+1})$,*

$$(2.6) \qquad \| U(t)x \| \leqq Me^{\omega|t|} \| x \|_k, \qquad t \in \mathbf{R}.$$

PROOF. To prove the assertion we need to introduce the Banach space $Z_\omega := C(\mathbf{R}; [D(A)])$ of all $D(A)$-valued functions on $\mathbf{R}$ which are $\| \cdot \|_1$-continuous, with finite

$$\sup\{e^{-\omega|t|} \| u(t) \|_1; t \in \mathbf{R}\}.$$

The norm of $u \in Z_\omega$ is this supremum.

Now we define a linear operator L on $[D(A^{k+1})]$ to Z_ω by

$$Ly := U(\cdot)y, \qquad y \in D(A^{k+1}).$$

To show that L is closed, let $y_n \to y$ $(n \to \infty)$ in $[D(A^{k+1})]$ and $Ly_n = U(\cdot)y_n \to z(\cdot)$ $(n \to \infty)$ in Z_ω. Then

$$(d^2/dt^2)U(t)y_n = AU(t)y_n \to Az(t) \qquad (n \to \infty)$$

uniformly in t on any finite subinterval of $\mathbf{R}$. Going to the limit in the equality

$$U(t)y_n = y_n + \int_0^t (t-s)(d^2/ds^2)U(s)y_n ds,$$

we have

$$z(t) = y + \int_0^t (t-s)Az(s)ds,$$

i.e., $z(t)$ is a solution to ACP on $D(A^{k+1})$. Since

$$\| z(t) \| + \| z''(t) \| = O(e^{\omega|t|}) \qquad \text{as } |t| \to \infty,$$

it follows from the uniqueness that $z(t) = U(t)y$. Therefore, L is closed. By the closed graph theorem we can find a constant $N > 0$ such that

$$\| U(t)y \|_1 \leqq Ne^{\omega|t|} \| y \|_{k+1}, \qquad y \in D(A^{k+1}).$$

Next let $b \in \rho(A)$ and $x \in D(A^{k+1})$. Then $y := R(b; A)x \in D(A^{k+2})$, and

$$\| y \|_{k+1} = \| R(b; A)x \|_{k+1} \leqq N_1 \| x \|_k,$$

where $N_1 = \max\{ \| AR(b; A) \|, \| R(b; A) \| \}$. On the other hand, it follows from Lemma 2.1 that $U(t)x = (b - A)U(t)y$. So, we have

$$\| U(t)x \| \leqq N_2 \| U(t)y \|_1,$$

where $N_2 = \max\{ |b|, 1\}$. Thus, we obtain (2.6) with $M = N \cdot N_1 \cdot N_2$. Q.E.D.

The next lemma is due to Oharu (see [10], Lemma 2.7).

LEMMA 2.3. *Let A be a closed linear operator with domain $D(A)$ dense in X and $\rho(A) \neq \emptyset$. Then for every pair of positive integers m and k with $m \geqq k$, $D(A^m)$ is dense in $[D(A^k)]$. In particular, $D(A^m)$ is a core for A, i.e., the closure of the restriction of A to $D(A^m)$ is again A for $m \in \mathbf{N}$.*

For a densely defined operator we have

PROPOSITION 2.4. *Let A be a closed linear operator with domain $D(A)$ dense in X and $\rho(A) \neq \emptyset$. Assume that ACP on $D(A^{k+1})$, $k \in \mathbf{N}$, is exponentially wellposed and let $\{U(t);\ t \in \mathbf{R}\}$ be the associated family of solution operators defined by (2.2). Then $U(t)$ can be uniquely extended to a continuous linear operator (the extension will be denoted again by $U(t)$) on $[D(A^k)]$ to X satisfying*

$$\| U(t)x \| \leqq Me^{\omega|t|} \| x \|_k, \qquad x \in D(A^k), \quad t \in \mathbf{R}, \tag{2.7}$$

$$U(t)x \in C(\mathbf{R}; X) \qquad \text{for } x \in D(A^k), \tag{2.8}$$

$$(d^2/dt^2)^j U(t)x = A^j U(t)x = U(t)A^j x, \qquad x \in D(A^{k+j}), \quad t \in \mathbf{R}, \tag{2.9}$$

$$\| (d/dt)U(t)x \| \leqq M|t| e^{\omega|t|} \| Ax \|_k, \qquad x \in D(A^{k+1}), \quad t \in \mathbf{R}. \tag{2.10}$$

PROOF. It follows from (2.6) that $U(t)$ has a bounded extension in $[D(A^k)]$. But, since $D(A^{k+1})$ is dense in $[D(A^k)]$ (see Lemma 2.3), the extension is unique and (2.7) follows from (2.6). Thus, we can prove (2.8).

Now we prove (2.9). We have only to show the second equality in the case of $j = 1$. The proof is based on the equality (2.4). Let $x \in D(A^{k+1})$. The proof is based on the equality (2.4). Let $x \in D(A^{k+1})$. Then, since $D(A^{k+2})$ is dense in $[D(A^{k+1})]$ (see Lemma 2.3), there is a sequence $\{x_n\}$ in $D(A^{k+2})$ such that $x_n \to x$ $(n \to \infty)$ in $[D(A^{k+1})]$. Hence it follows that $x_n \to x$ and $Ax_n \to Ax$ $(n \to \infty)$ in $[D(A^k)]$. By virtue of (2.7) we have

$$U(t)x_n \to U(t)x \quad \text{and} \quad U(t)Ax_n \to U(t)Ax \quad \text{in } X.$$

Going to the limit $n \to \infty$ in (2.4) with x replaced by x_n, we see from the closedness of A that $U(t)x \in D(A)$ and $AU(t)x = U(t)Ax$, $x \in D(A^{k+1})$. Thus we obtain (2.9) with $j = 1$. The rest of the proof is easily completed by induction.

Now that $(d/dt)U(0)x = u'(0; x, 0) = 0$, we see from (2.9) that

$$(d/dt)U(t)x = \int_0^t U(s)Axds, \qquad x \in D(A^{k+1}).$$

Therefore, (2.10) follows from (2.7). Q.E.D.

Let Z be another Banach space. Then we denote by $B(Z, X)$ the set of all bounded linear operators on Z to X.

DEFINITION 2.5. $\{U(t); t \in \mathbf{R}\} \subset B([D(A^k)], X)$ constructed in Proposition 2.4 is called *a family of generalized solution operators* for ACP.

In the remainder of this section we shall mention certain classes of semigroups of bounded linear operators on X.

Let $B(X)$ be the set of all bounded linear operators on X to X. Then a one-parameter family $\{T(t); t > 0\}$ in $B(X)$ is called a *semigroup* on X if $T(t+s) = T(t)T(t)$ for $t, s > 0$ and if $T(t)$ is strongly continuous for $t > 0$. We denote by A_0 the *infinitesimal generator* of $\{T(t)\}$, i.e.,

$$A_0 x = \lim_{h \to +0} h^{-1}[T(h) - I]x$$

whenever the limit exists. If A_0 is closable, then the closure of A_0 is called the *complete infinitesimal generator* of $\{T(t)\}$.

Now, let $\{T(t); t > 0\}$ be a semigroup on X. If in particular $T(t)$ can be extended holomorphically into a sector $|\arg t| < \alpha$ for some $0 < \alpha \leqq \pi/2$, then $\{T(t); |\arg t| < \alpha\}$ is called a *holomorphic semigroup* on X.

Let ω_0 be the type of $\{T(t)\}$: $\omega_0 = \lim_{t \to \infty} t^{-1} \log \| T(t) \|$. Let m be a nonnegative integer and $\lambda \in \mathbf{C}$ with $\operatorname{Re} \lambda > \omega_0$. Then we define

$$\begin{aligned} R_m(\lambda)v &:= (1/m!) \int_0^\infty t^m e^{-\lambda t} T(t) v\, dt \\ &= (1/m!) \lim_{\substack{\delta \to +0 \\ \tau \to \infty}} \int_\delta^\tau t^m e^{-\lambda t} T(t) v\, dt \end{aligned}$$

whenever the limit exists.

DEFINITION 2.6. Let $n \in \mathbf{N}$. Then a holomorphic semigroup $\{T(t); |\arg t| < \alpha\}$ on X is said to be of class (H_n) if it satisfies the following conditions:

(i) $T(t)x = 0$ for all $t > 0$ implies that $x = 0$,

(ii) $X_0 := \bigcup_{t>0} T(t)[X]$ is dense in X,

(iii) for each ε $(0 < \varepsilon \leqq \alpha)$ there exists $L_\varepsilon > 0$ such that

$$\| T(t) R_{n-1}(b) x \| \leqq L_\varepsilon e^{\omega \operatorname{Re} t} \| x \|, \qquad x \in X_0, \quad |\arg t| \leqq \alpha - \varepsilon,$$

where b and ω are constants satisfying

$$b > \omega > \omega_0 = \lim_{t\to\infty} t^{-1} \log \| T(t) \|,$$

(iv) there is a constant $K > 0$ such that

$$\| R_0(b)x \| \leqq K \| x \| \qquad \text{for } x \in X_0.$$

REMARK 2.7. A semigroup of class (H_n) was introduced in [11], however, Definition 2.6 is an equivalent definition given in the Appendix of this paper.

3. Abstract Cauchy problems

Let A be a densely defined closed linear operator in X with $\rho(A) \neq \emptyset$. Suppose that ACP on $D(A^{k+1})$, $k \in \mathbf{N}$, is exponentially wellposed with the constant of exponential growth $\omega > 0$. Then we have constructed a family $\{U(t);\ t \in \mathbf{R}\}$ of generalized solution operators (see Definition 2.5). Let $x \in D(A^k)$ and $\lambda \in \mathbf{C}$ with $\operatorname{Re} \lambda > \omega$. In view of (2.7) we see that

$$L_0(\lambda^2)x := \frac{1}{\lambda} \int_0^\infty e^{-\lambda t} U(t) x dt \tag{3.1}$$

is well defined, i.e., $\lambda L_0(\lambda^2)x$ is the Laplace transform of $U(t)x$.

LEMMA 3.1. *Let* $\operatorname{Re}\lambda > \omega$. *Then there is a constant* $K > 0$ *such that*

$$\| L_0(\lambda^2)x \| \leqq K \| x \|, \qquad x \in D(A^k). \tag{3.2}$$

PROOF. Let $x \in D(A^k)$ and $b \in \rho(A)$. To simplify the notation let us introduce the operator

$$V_j^k(t)x := (b - A)^j U(t) R(b; A)^k x, \qquad 0 \leqq j \leqq k.$$

Since $U(t)x = V_k^k(t)x = (b - d^2/dt^2) V_{k-1}^k(t)x$, it follows from (3.1) that

$$\begin{aligned}\lambda L_0(\lambda^2)x &= b \int_0^\infty e^{-\lambda t} V_{k-1}^k(t)xdt - \int_0^\infty e^{-\lambda t}(d^2/dt^2) V_{k-1}^k(t)xdt \\ &= (b - \lambda^2) \int_0^\infty e^{-\lambda t} V_{k-1}^k(t)xdt + \lambda R(b; A)x,\end{aligned}$$

where we have used (2.10). Continuing such a calculation, we obtain

$$\lambda L_0(\lambda^2)x = (b-\lambda^2)^k \int_0^\infty e^{-\lambda t} V_0^k(t)xdt + \sum_{j=0}^{k-1} \lambda (b-\lambda^2)^j R(b; A)^{j+1} x.$$

Since $V_0^k(t) = U(t)R(b; A)^k$ is bounded on X by (2.7), we can obtain (3.2).

Q.E.D.

LEMMA 3.2. *Let* $\operatorname{Re}\lambda > \omega$. *Then* $\lambda^2 \in \rho(A)$ *and*

$$L_0(\lambda^2)x = R(\lambda^2; A)x = (\lambda^2 - A)^{-1}x, \qquad x \in D(A^k).$$

PROOF. Let $L(\lambda^2) \in B(X)$ be the extension of $L_0(\lambda^2)$; note that $D(A^k)$ is dense in X. Then it suffices to show that

$$L(\lambda^2)(\lambda^2 - A)x = x, \qquad x \in D(A), \tag{3.3}$$

$$(\lambda^2 - A)L(\lambda^2)y = y, \qquad y \in X. \tag{3.4}$$

First we prove (3.3). Let $x \in D(A^{k+1})$. Then we see from (2.9) that

$$\begin{aligned} L(\lambda^2)(\lambda^2 - A)x &= L_0(\lambda^2)(\lambda^2 - A)x \\ &= \frac{1}{\lambda}\int_0^\infty e^{-\lambda t}(\lambda^2 - d^2/dt^2)U(t)x\,dt = x. \end{aligned}$$

Since $D(A^{k+1})$ is a core for A (see Lemma 2.3), we can obtain (3.3).

Next let $y \in D(A^{k+1})$. Then, as shown above, we have

$$\int_0^\infty e^{-\lambda t}(\lambda^2 - A)U(t)y\,dt = \lambda y.$$

It follows from the closedness of $\lambda^2 - A$ that

$$(\lambda^2 - A)L_0(\lambda^2)y = y.$$

Since $D(A^{k+1})$ is dense in X, (3.4) follows again from the closedness of $\lambda^2 - A$.

Q.E.D.

Now we are in a position to state a characterization theorem for the exponential wellposedness of ACP on $D(A^{k+1})$, $k \in \mathbf{N}$.

THEOREM 3.3. *Let A be a densely defined closed linear operator in X with nonempty resolvent set $\rho(A)$. Let $\omega > 0$ be the constant of exponential growth. Then ACP on $D(A^{k+1})$, $k \in \mathbf{N}$, is exponentially wellposed if and only if A satisfies the following two conditions*:

(I) *there is* $\omega > 0$ *such that* $\rho(A) \supset \{\lambda^2 \in \mathbf{C}; \operatorname{Re}\lambda > \omega\}$,

(II) *there is a constant* $M > 0$ *such that for* $x \in D(A^k)$,

$$\|(d/d\lambda)^{n-1}[\lambda R(\lambda^2; A)x]\| \leq M(n-1)!\,(\operatorname{Re}\lambda - \omega)^{-n}\,\|x\|_k, \qquad \operatorname{Re}\lambda > \omega, \quad n \in \mathbf{N}.$$

PROOF. First we prove the "only if" part. It follows from Lemma 3.2 and Proposition 2.4 that

$$\{\lambda^2 \in \mathbf{C}; \operatorname{Re} \lambda > \omega\} \subset \rho(A)$$

and for $x \in D(A^k)$, $\lambda R(\lambda^2; A)$ and its n-th derivative are given by

$$\lambda R(\lambda^2; A)x = \int_0^\infty e^{-\lambda t} U(t)x dt,$$

$$(d/d\lambda)^n [\lambda R(\lambda^2; A)x] = \int_0^\infty (-t)^n e^{-\lambda t} U(t)x dt, \qquad n \in \mathbf{N}.$$

Therefore, condition (II) follows from (2.7).

The "if" part has already been proved in Takenaka–Okazawa [16] (see Theorem 3.4 below). Q.E.D.

Here for later use we state a better sufficient condition for ACP on $D(A^{k+1})$ to be exponentially wellposed. The conditions are regarded as a real version for conditions (I) and (II) in Theorem 3.3.

THEOREM 3.4. *Let A be a densely defined and closed linear operator in X, and let $k \in \mathbf{N}$. Assume that*

($\mathrm{I}_{\mathrm{real}}$) *there is a constant $\omega > 0$ such that*

$$\rho(A) \supset \{\lambda^2 \in \mathbf{R}; \lambda > \omega\},$$

($\mathrm{II}_{\mathrm{real}}$) *there is a constant $M > 0$ such that for $x \in D(A^k)$,*

$$\| (d/d\lambda)^{n-1} [\lambda R(\lambda^2; A)x] \| \leqq M(n-1)!(\lambda - \omega)^{-n} \| x \|_k, \qquad \lambda > \omega, \quad n \in \mathbf{N}.$$

Then there is a family $\{C(t); t \in \mathbf{R}\}$ of (generalized solution) operators given by

$$C(t)x = \lim_{n \to \infty} \frac{(-1)^{n-1}}{(n-1)!} \left(\frac{n}{t}\right)^n \left(\frac{d}{d\lambda}\right)^{n-1} [\lambda R(\lambda^2; A)x] \big|_{\lambda = n/t} \qquad (t > 0),$$

$$C(0)x = x, \quad C(t)x = C(-t)x \qquad (t < 0) \quad \textit{for } x \in D(A^k),$$

where the convergence is uniform with respect to t on every bounded interval of $[0, \infty)$. *$\{C(t)\}$ has the following properties*:

(a) *for each $x \in D(A^k)$, $C(t)x$ is continuous on* $\mathbf{R}$,

(b) $\| C(t)x \| \leqq M e^{\omega |t|} \| x \|_k$, $x \in D(A^k)$, $t \in \mathbf{R}$,

(c) $(d^2/dt^2)^j C(t)x = A^j C(t)x = C(t)A^j x$, $x \in D(A^{k+j})$, $t \in \mathbf{R}$,

(d) $\| (d/dt)C(t)x \| \leqq M |t| e^{\omega |t|} \| x \|_k$, $x \in D(A^k)$, $t \in \mathbf{R}$,

(e) $\lambda R(\lambda^2; A)x = \int_0^\infty e^{-\lambda t} C(t)x dt$, $x \in D(A^k)$, $\lambda > \omega$,

(f) *$u(t; x, y) = C(t)x + \int_0^t C(s)y ds$ is a unique solution to ACP on $D(A^{k+1})$ with $\| u(t; x, y) \| \leqq M(1 + |t|)e^{\omega |t|}(\| x \|_k + \| y \|_k)$, and*

$$\| (d/dt)u(t;x,y) \| \leqq M(1+|t|)e^{\omega|t|}(\| Ax \|_k + \| y \|_k) \qquad \textit{for } x, y \in D(A^{k+1}).$$

Consequently, ACP on $D(A^{k+1})$ is exponentially wellposed.

The properties (c), (d) and (f) are not explicitly stated but essentially proved in [16].

Corollary 3.5. *Let A and $\{C(t); t\in\mathbf{R}\}$ be as in Theorem 3.4. Then for $x\in D(A^k)$ and $t>0$, $C(t)x$ is also given by*

$$C(t)x = \lim_{n\to\infty} \frac{(-1)^n}{n!}\left(\frac{n}{t}\right)^{n+1}\left(\frac{d}{d\lambda}\right)^n [\lambda R(\lambda^2;A)x]\,|_{\lambda=n/t}. \tag{3.5}$$

Proof. Let us use the following notations:

$$F_\lambda^n x := (d/d\lambda)^n[R(\lambda^2;A)x], \qquad G_\lambda^n x := (d/d\lambda)^n[\lambda R(\lambda^2;A)x].$$

Then we obtain a simple relation

$$\lambda G_\lambda^n x + nG_\lambda^{n-1}x = AF_\lambda^n x. \tag{3.6}$$

Furthermore, ($\mathrm{II}_{\mathrm{real}}$) implies that for $x\in D(A^k)$ and $\lambda>\omega$

$$\| F_\lambda^n x \| \leqq M(n+1)!(\lambda-\omega)^{-n-2}\| x \|_k. \tag{3.7}$$

(For proofs of (3.6) and (3.7), see [16].) Now set

$$C_n(t)x := \frac{(-1)^{n-1}}{(n-1)!}(n/t)^n G_{n/t}^{n-1}x, \quad t>0,$$

$$W_n(t) := \frac{(-1)^n}{n!}(n/t)^{n+1}G_{n/t}^n, \qquad t>0.$$

To prove (3.5), it suffices by Theorem 3.4 to show that

$$\lim_{n\to\infty} \| C_n(t)x - W_n(t)x \| = 0, \quad x\in D(A^k), \quad t>0.$$

But we see from (3.6) that

$$C_n(t)x - W_n(t)x = \frac{(-1)^{n-1}}{n!}(n/t)^n AF_{n/t}^n x.$$

Therefore, it follows from (3.7) that for $x\in D(A^{k+1})$,

$$\| C_n(t)x - W_n(t)x \| \leqq \frac{Mt^2}{n}\left(1 + \frac{1}{n}\right)\left(1 - \frac{\omega t}{n}\right)^{-n-2} \| Ax \|_k.$$

Noting that $\| W_n(t)x \| \leqq M(1 - \omega t/n)^{-n-1} \| x \|_k$, $x \in D(A^k)$ (use condition ($\mathrm{II}_{\mathrm{real}}$)), (3.5) can be obtained by Lemma 2.3. Q.E.D.

REMARK 3.6. Let $\{C(t)\}$ be as in Theorem 3.4. Then for $x \in D(A^{2k})$, $\{C(t)x\}$ has the cosine property

$$C(t+s)x + C(t-s)x = 2C(t)C(s)x, \qquad t, s \in \mathbf{R};$$

see [16]. Consequently, if we take $k = 0$ in Theorem 3.4, then $\{C(t)\}$ forms a cosine family. The generation theorem of cosine families was established by Sova [15], Da Prato–Giusti [2] and Fattorini [3]. Here we note that Sova and Fattorini proved (3.5) for a cosine family $\{C(t)\}$ through the Post–Widder inversion formula.

Now let B be a densely defined closed linear operator in X with nonempty resolvent set: $\rho(B) \neq \varnothing$. We consider the first order differential equation:

$$v'(t) = Bv(t), \qquad t \in \mathbf{R}, \quad v(0) = x. \tag{3.8}$$

Applying the Oharu–Sanekata theorem to $\pm B$, it is easily seen that the problem (3.8) on $D(B^{k+1})$, $k \in \mathbf{N}$, is exponentially wellposed if and only if there are constants $\omega \geqq 0$ and $M > 0$ such that

$$\{\lambda \in \mathbf{R}; |\lambda| > \omega\} \subset \rho(B), \tag{3.9}$$

$$\| (\lambda - B)^{-n} \| \leqq M(|\lambda| - \omega)^{-n} \| x \|_k, \qquad |\lambda| > \omega, \quad n \in \mathbf{N}. \tag{3.10}$$

COROLLARY 3.7. *Let B be a densely defined closed linear operator in X, satisfying* (3.9) *and* (3.10). *Put $A = B^2$ in ACP for the second order equation. Then ACP on $D(A^{k+1}) = D(B^{2(k+1)})$ is exponentially wellposed.*

PROOF. Since we have

$$\tfrac{1}{2}[(\lambda - B)^{-1} + (\lambda + B)^{-1}] = \lambda(\lambda^2 - B^2)^{-1},$$

we can easily show that $A = B^2$ satisfies the assumption of Theorem 3.4.

Q.E.D.

PROPOSITION 3.8. *Let A be a densely defined and closed linear operator in X satisfying*

(I) *there is $\omega > 0$ such that $\{\lambda^2 \in \mathbf{C}; \operatorname{Re} \lambda > \omega\} \subset \rho(A)$,*

($\mathrm{II}_{\mathrm{pol}}$) *there are constants* $M > 0$ *and* $\delta > 0$ *such that*

$$\| \lambda R(\lambda^2; A) \| \leqq M(1 + |\lambda|^{2\delta}), \qquad \operatorname{Re} \lambda > \omega.$$

Let $\beta > \max\{0, \omega\}$ *and* $k := [\delta + 1/2] + 1$. *Then there is a constant* $N > 0$ *such that for* $x \in D(A^k)$, $\operatorname{Re} \lambda > \beta$ *and* $n \in \mathbf{N}$,

$$\| (d/d\lambda)^{n-1}[\lambda R(\lambda^2; A)x] \| \leqq N(n-1)!(\operatorname{Re} \lambda - \beta)^{-n} \| x \|_k. \tag{3.11}$$

Consequently, ACP on $D(A^{k+1})$ *is exponentially wellposed.*

PROOF. Let $\lambda^2 \in \rho(A)$ and $x \in D(A^k)$. Then we have

$$R(\lambda^2; A)x = \sum_{j=1}^{k} \lambda^{-2j} A^{j-1} x + \lambda^{-2k} R(\lambda^2; A) A^k x. \tag{3.12}$$

Let $\beta > \max\{0, \omega\}$. Then we can define the inverse Laplace transform of $\lambda R(\lambda^2; A)x$:

$$z(t; x) := \lim_{\tau \to \infty} (2\pi i)^{-1} \int_{\beta - i\tau}^{\beta + i\tau} \lambda e^{\lambda t} R(\lambda^2; A) x d\lambda, \qquad t > 0.$$

By calculation of residues and condition ($\mathrm{II}_{\mathrm{pol}}$) we have

$$z(t; x) = \sum_{j=1}^{k} \frac{t^{2j-2}}{(2j-2)!} A^{j-1} x + (2\pi i)^{-1} \int_{\beta - i\infty}^{\beta + i\infty} \lambda^{-2k+1} e^{\lambda t} R(\lambda^2; A) A^k x d\lambda.$$

The second term on the right-hand side is estimated by

$$\frac{M}{\pi} e^{\beta t} \int_0^{\infty} \frac{1 + (\beta^2 + \tau^2)^{\delta}}{(\beta^2 + \tau^2)^k} d\tau \, \| A^k x \| ;$$

note that $k - \delta > 1/2$. Setting $z(0; x) = x$, we can find a constant $N > 0$ such that

$$\| z(t; x) \| \leqq N e^{\beta t} \| x \|_k. \tag{3.13}$$

Now we consider the Laplace transform of $z(t; x)$. Noting that

$$\begin{aligned} \int_0^{\infty} \mathrm{e}^{-\lambda t} & \int_{\beta - i\infty}^{\beta + i\infty} \mu^{-2k+1} e^{\mu t} R(\mu^2; A) A^k x d\mu dt \\ &= \int_{\beta - i\infty}^{\beta + i\infty} (\lambda - \mu)^{-1} \mu^{-2k+1} R(\mu^2; A) A^k x d\mu \\ &= (2\pi i) \lambda^{-2k+1} R(\lambda^2; A) A^k x, \end{aligned}$$

we see from (3.12) that

$$\lambda R(\lambda^2; A)x = \int_0^\infty e^{-\lambda t} z(t; x)dt$$

and hence

$$(d/d\lambda)^{n-1}[\lambda R(\lambda^2; A)x] = \int_0^\infty (-t)^{n-1} e^{-\lambda t} z(t; x)dt.$$

Thus, (3.11) follows from (3.13). Q.E.D.

REMARK 3.9. Proposition 3.8 is a second-order analogue of Theorem 4.7(a) in Oharu [10] and was first proved by Cioranescu ([1], Corollary after Proposition) with slightly rough estimates (see also Neubrander [9], Corollary 7.8).

Anyway, condition ($\mathrm{II}_{\mathrm{pol}}$) may be replaced by

($\mathrm{II}_{\mathrm{pol}}$)′ for any $\varepsilon > 0$ there is a constant $M_\varepsilon > 0$ such that

$$\| \lambda R(\lambda^2; A) \| \leqq M_\varepsilon(1 + |\lambda|^{2\delta}), \qquad \operatorname{Re}\lambda \geqq \omega + \varepsilon.$$

Before concluding this section we give an example of an operator satisfying conditions (I) and ($\mathrm{II}_{\mathrm{pol}}$)′. Let E be a $(n+1)\times(n+1)$ unit matrix and F be a $(n+1)\times(n+1)$ nilpotent matrix such that only the first upper diagonal elements are equal to one.

EXAMPLE 3.10. Let $X = \Pi^{n+1} \mathbf{L}^2(\mathbf{R})$ and consider the multiplication operator in X defined by

$$(Ax)(p) = A(p)x(p) = \sum_{j=1}^{n+1} (-p^2)^j F^{j-1} x(p), \qquad p \in \mathbf{R},$$

where $x(p) = {}^{\mathrm{T}}(x_1(p), x_2(p), \ldots, x_{n+1}(p))$ and $F^0 = E$. Let $\zeta \in \mathbf{C}$ with $|\arg\zeta| < \pi$. Then the matrix $\zeta E - A(p)$ is invertible and the norm of its inverse

$$\begin{aligned} &[\zeta E - A(p)]^{-1} \\ &= (p^2+\zeta)^{-1}E + \sum_{j=1}^{n} (-\zeta)^{j-1} p^{2(j+1)} (p^2+\zeta)^{-(j+1)} F^j \end{aligned} \tag{3.14}$$

is uniformly bounded in p. This shows that $\rho(A)$ contains the sector $|\arg\zeta| < \pi$, i.e., $\{\lambda^2 \in \mathbf{C}; \operatorname{Re}\lambda > 0\} \subset \rho(A)$. Now let $\varepsilon > 0$ and $\lambda \in \mathbf{C}$ with $\operatorname{Re}\lambda \geqq \varepsilon$. Then, since

$$\lambda^2 = (\operatorname{Re}\lambda)^2 - (\operatorname{Im}\lambda)^2 + 2i(\operatorname{Re}\lambda)(\operatorname{Im}\lambda),$$

we see that for $p \in \mathbf{R}$,

$$|p^2+\lambda^2| \geqq |\mathrm{Im}\,\lambda^2| \geqq 2\varepsilon|\mathrm{Im}\,\lambda| \geqq \sqrt{2}\varepsilon|\lambda| \quad \text{if } |\arg\lambda| \geqq \pi/4,$$

$$|p^2+\lambda^2| \geqq |\lambda|^2 \geqq \varepsilon|\lambda| \qquad \text{if } |\arg\lambda| \leqq \pi/4.$$

Therefore, it follows from (3.14) that for $\mathrm{Re}\,\lambda \geqq \varepsilon$,

$$\begin{aligned}\|(\lambda^2-A)^{-1}\| &\leqq \sup_{p\in\mathbf{R}}\Big(|p^2+\lambda^2|^{-1}+\sum_{j=1}^{n}|\lambda|^{2(j-1)}p^{2(j+1)}|p^2+\lambda^2|^{-(j+1)}\Big)\\ &\leqq (\varepsilon|\lambda|)^{-1}+\sum_{j=1}^{n}|\lambda|^{2(j-1)}[1+|\lambda|^2(\varepsilon|\lambda|)^{-1}]^{j+1}.\end{aligned}$$

So there is a constant $M_\varepsilon > 0$ such that

$$\|\lambda(\lambda^2-A)^{-1}\| \leqq M_\varepsilon(1+|\lambda|^{3n}), \qquad \mathrm{Re}\,\lambda \geqq \varepsilon.$$

Since $D(A)$ is dense in X, A satisfies the assumption of Proposition 3.8 with $(\mathrm{II}_{\mathrm{pol}})$ replaced by $(\mathrm{II}_{\mathrm{pol}})'$.

REMARK 3.11. It is shown in [11] that the operator in Example 3.10 is the complete infinitesimal generator of a semigroup of class (H_n).

4. The abstract Weierstrass formula

Let A be a densely defined and closed linear operator in X. Assume that A satisfies two conditions $(\mathrm{I}_{\mathrm{real}})$ and $(\mathrm{II}_{\mathrm{real}})$ in Theorem 3.4. Then there is a family $\{C(t);\, t\in\mathbf{R}\}$ of generalized solution operators for ACP. The properties of $\{C(t)\}$ are summarized in Theorem 3.4.

Now we introduce a new family $\{T(t);\, \mathrm{Re}\,t>0\}$ in $B(X)$. But $T(t)x$ is first defined on $D(A^k)$ by the abstract Weierstrass formula

$$T(t)x = \frac{1}{\sqrt{\pi t}}\int_0^\infty e^{-s^2/4t}C(s)x\,ds \qquad \text{for } x\in D(A^k). \tag{4.1}$$

As is easily seen, $T(t)x$ is holomorphic in t in the right half plane. First we note that $T(t)x$ $(t>0)$ has exponential growth. In fact, setting $s = 2\sqrt{t}\sigma$ in (4.1), we have

$$T(t)x = \frac{2}{\sqrt{\pi}}\int_0^\infty \exp(-\sigma^2)C(2\sqrt{t}\sigma)x\,d\sigma.$$

Thus, we see from Theorem 3.4(a) that

$$(4.2) \qquad \| T(t)x \| \leqq 2M \exp(\omega^2 t) \| x \|_k, \qquad x \in D(A^k), \quad t > 0.$$

In what follows we shall show that $\{T(t); \operatorname{Re} t > 0\}$ is a holomorphic semigroup on X (see Proposition 4.5) such that for any ε $(0 < \varepsilon < \pi/2)$, $\{T(t); |\arg t| \leqq (\pi/2) - \varepsilon\}$ forms a semigroup of class (H_k) introduced in Section 2 (see Proposition 4.11).

LEMMA 4.1. *Let $\{C(t)\}$ be a family of generalized solution operators for ACP. Let $x \in D(A^k)$. Then $\{T(t)\}$ defined by (4.1) has the following properties*:

(a) $\| T(t)x \| \leqq 2M(|t|/\operatorname{Re} t)^{1/2} \exp(\omega^2 |t|^2/\operatorname{Re} t) \| x \|_k$, $\operatorname{Re} t > 0$,
(b) $\| T(t)x \| \leqq (2M/\sqrt{\sin \varepsilon}) \exp(\omega^2 \operatorname{Re} t/\sin^2 \varepsilon) \| x \|_k$, $|\arg t| \leqq \pi/2 - \varepsilon$,
(c) $\| T(t)x - x \| \to 0$ as $|t| \to 0$ with $|\arg t| \leqq \pi/2 - \varepsilon$,
(d) $R(\lambda; A)x = \int_0^\infty e^{-\lambda t} T(t)x dt$, $\lambda > \omega^2$.

PROOF. (a) Let $x \in D(A^k)$ and $\operatorname{Re} t > 0$. Then it follows from Theorem 3.4(b) that

$$\begin{aligned} \| T(t)x \| &\leqq (\pi |t|)^{-1/2} \int_0^\infty \exp\left(-\frac{\operatorname{Re} t}{4|t|^2} s^2 \right) \| C(s)x \| ds \\ &\leqq 2M \left(\frac{|t|}{\pi \operatorname{Re} t} \right)^{1/2} \int_0^\infty \exp\left(-\sigma^2 + \frac{2\omega |t|}{\sqrt{\operatorname{Re} t}} \sigma \right) \| x \|_k d\sigma \\ &= 2M \left(\frac{|t|}{\operatorname{Re} t} \right)^{1/2} \exp\left(\frac{\omega^2 |t|^2}{\operatorname{Re} t} \right) \| x \|_k. \end{aligned}$$

(b) Given ε $(0 < \varepsilon < \pi/2)$, we have

$$(4.3) \qquad \operatorname{Re} t = |t| \cos(\arg t) \geqq |t| \sin \varepsilon \; (|\arg t| \leqq \pi/2 - \varepsilon).$$

Thus, the desired estimate follows from (a).

(c) Noting that

$$(\pi t)^{-1/2} \int_0^\infty \exp\left(-\frac{s^2}{4t} \right) ds = 1, \qquad \operatorname{Re} t > 0,$$

we obtain

$$\| T(t)x - x \| \leqq (\pi |t|)^{-1/2} \int_0^\infty \exp\left(-\frac{\operatorname{Re} t}{4|t|^2} s^2 \right) \| C(s)x - x \| ds.$$

Since $C(s)x$ is continuous on $\mathbf{R}$, for any $\eta > 0$ there is $\delta = \delta(\eta)$ such that

$$\| C(s)x - x \| < \eta \qquad (|s| < \delta).$$

Then we divide the integral into two parts:

$$\| T(t)x - x \| \leqq (\pi |t|)^{-1/2}\left(\int_0^\delta + \int_\delta^\infty\right) = I_1 + I_2.$$

Fix ε $(0 < \varepsilon < \pi/2)$. Then it follows from (4.3) that

$$I_1 \leqq \eta(\pi |t|)^{-1/2} \int_0^\delta \exp\left(-\frac{\sin \varepsilon}{4|t|} s^2\right) ds < \frac{\eta}{\sqrt{\sin \varepsilon}}.$$

Next, setting $N = (M+1)(\| x \|_k + \| x \|)$, we have

$$I_2 \leqq N(\pi |t|)^{-1/2} \int_\delta^\infty \exp\left(-\frac{\sin \varepsilon}{4|t|} s^2 + \omega s\right) ds$$

$$= \frac{2N}{\sqrt{\pi \sin \varepsilon}} \int_{\gamma(t)}^\infty \exp(-\sigma^2) d\sigma \exp\left(\frac{\omega^2 |t|}{\sin \varepsilon}\right),$$

where $\gamma(t) = (\delta\sqrt{\sin \varepsilon}/2|t|) - \omega(|t|/\sin \varepsilon)^{1/2}$. Since $\gamma(t) \to \infty$ as $|t| \to 0$ with $|\arg t| \leqq \pi/2 - \varepsilon$, we can obtain (c).

(d) We give a formal calculation. Let $x \in D(A^k)$ and $\lambda > \omega^2$. Then the assertion follows from (4.1):

$$\int_0^\infty e^{-\lambda t} T(t)x dt = \int_0^\infty e^{-\lambda t}\left[(\pi t)^{-1/2} \int_0^\infty e^{-s^2/4t} C(s)x ds\right] dt$$

$$= \int_0^\infty \left[\int_0^\infty (\pi t)^{-1/2} \exp\left(-\frac{s^2}{4t} - \lambda t\right) dt\right] C(s)x ds$$

$$= \lambda^{-1/2} \int_0^\infty e^{-\sqrt{\lambda} s} C(s)x ds = R(\lambda; A)x,$$

where we have used Theorem 3.4(e). The calculation can be easily justified.

Q.E.D.

Lemma 4.2. *Let $\{C(t)\}$ be as in Lemma* 4.1. *Let $y \in D(A^{k+1})$. Then $C(t)x$ satisfies the following equality*:

$$\int_0^\infty \exp\left(-\frac{s^2}{4t}\right)(d^2/ds^2)C(s)yds$$

$$(4.4)\qquad = \frac{1}{2t}\int_0^\infty \left(\frac{s^2}{2t}-1\right)\exp\left(-\frac{s^2}{4t}\right)C(s)yds$$

$$= \frac{1}{2t}\int_0^\infty (d^2/dr^2)\left[\exp\left(-\frac{r^2}{2}\right)\right]\bigg|_{r=s/\sqrt{2t}} C(s)yds, \quad \operatorname{Re} t > 0.$$

PROOF. We see from Theorem 3.4(d) that

$$2t\int_0^\infty \exp\left(-\frac{s^2}{4t}\right)(d^2/ds^2)C(s)yds = \int_0^\infty s\cdot\exp\left(\cdot-\frac{s^2}{4t}\right)(d/ds)C(s)yds.$$

One more integration by parts gives (4.4) because of Theorem 3.4(b). Q.E.D.

LEMMA 4.3. *Let $\{C(t)\}$ and $\{T(t)\}$ be as in Lemma* 4.1. *Then $\{T(t)\}$ has the following properties*:

(a) $(d/dt)T(t)x = AT(t)x = T(t)Ax$, $x \in D(A^{k+1})$, $\operatorname{Re} t > 0$,

(b) $A^jT(t)x = T(t)A^jx$, $x \in D(A^{k+j})$, $j \in \mathbf{N}$, $\operatorname{Re} t > 0$,

(c) $T(t+s)x = T(t)T(s)x$, $x \in D(A^{2k})$, $\operatorname{Re} t > 0$, $\operatorname{Re} s > 0$.

PROOF. (a) Differentiating (4.1) in t, we have

$$\frac{d}{dt}T(t)x = (\pi t)^{-1/2}\frac{1}{2t}\int_0^\infty \left(\frac{s^2}{2t}-1\right)\exp\left(-\frac{s^2}{4t}\right)C(s)xds.$$

It follows from (4.4) and Theorem 3.4(c) that

$$(4.5)\qquad \begin{aligned}\frac{d}{dt}T(t)x &= (\pi t)^{-1/2}\int_0^\infty \exp\left(-\frac{s^2}{4t}\right)(d^2/ds^2)C(s)xds\\ &= (\pi t)^{-1/2}\int_0^\infty \exp\left(-\frac{s^2}{4t}\right)AC(s)xds.\end{aligned}$$

Since A is closed, we obtain (a).

(b) is shown by induction. The case of $i = 1$ is contained in (a).

(c) It suffices by the analyticity to show the "semigroup" property for $t, s > 0$. Let $x \in D(A^{2k+1})$. Then it follows from (4.2) and (a) that $T(t-r)T(s+r)x$ is continuously differentiable in r. Hence we see from Lemma 4.1(c) that

$$T(t+s)x - T(t)T(s)x = \int_0^t \frac{d}{dr}[T(t-r)T(s+r)x]dr$$
$$= \int_0^t T(t-r)(A-A)T(s+r)xdr = 0.$$

Here we note that $T(t)T(s) \in B([D(A^{2k})], X)$. Since $D(A^{2k+1})$ is dense in $[D(A^{2k})]$ (see Lemma 2.3), we obtain (c). Q.E.D.

LEMMA 4.4. *Let $\{T(t)\}$ be as in Lemma 4.1. Let $t \in \mathbf{C}$ with $|\arg t| \leqq (\pi/2) - \varepsilon$ $(0 < \varepsilon \leqq \pi/2)$. Then there are polynomials $p(|t|)$ and $q(|t|)$ with non-negative coefficients depending on ε such that*

$$\text{(4.6)} \qquad \| T(t)x \| \leqq N|t|^{-k}p(|t|)\exp\left(\frac{\omega^2 \operatorname{Re} t}{\sin^2 \varepsilon}\right) \|x\|, \quad x \in D(A^k),$$

$$\text{(4.7)} \quad \left\| \frac{d}{dt} T(t)x \right\| \leqq N|t|^{-(k+1)}q(|t|)\exp\left(\frac{\omega^2 \operatorname{Re} t}{\sin^2 \varepsilon}\right) \|x\|, \quad x \in D(A^{k+1}),$$

where $N = (2/\pi)^{1/2}M \sum_{j=0}^k \| A^j R(b;A)^k \|$ and the degrees of $p(|t|)$ and $q(|t|)$ are k and $k+1$, respectively.

PROOF. First let $x \in D(A^k)$ and $b \in \rho(A)$ $(b > \omega^2)$. As in the proof of Lemma 3.1 we set

$$V_j^k(s)x := (b-A)^j C(s)R(b;A)^k x, \qquad 0 \leqq j \leqq k.$$

Since $C(s)x = (b - d^2/ds^2)V_{k-1}^k(s)x$, it follows from (4.4) that

$$(\pi t)^{1/2}T(t)x = \int_0^\infty \left(b - \frac{1}{2t}\frac{d^2}{dr^2}\right)\left[\exp\left(-\frac{r^2}{2}\right)\right]\Bigg|_{r=s/\sqrt{2t}} V_{k-1}^k(s)xds$$
$$= \int_0^\infty \left(b - \frac{1}{2t}\frac{d^2}{dr^2}\right)^k\left[\exp\left(-\frac{r^2}{2}\right)\right]\Bigg|_{r=s/\sqrt{2t}} V_0^k(s)xds$$
$$= \int_0^\infty \sum_{j=0}^k \binom{k}{j} b^{k-j}\left(\frac{-1}{2t}\right)^j\left(\frac{d^2}{dr^2}\right)^j\left[\exp\left(-\frac{r^2}{2}\right)\right]\Bigg|_{r=s/\sqrt{2t}} V_0^k(s)xds,$$

where $\binom{k}{j}$ is the binomial coefficient. Remembering that the Hermite polynomial $H_{2j}(r)$ is defined by

$$H_{2j}(r) = \exp(r^2/2)(d^2/dr^2)^j[\exp(-r^2/2)], \qquad j \in \mathbf{N},$$

we obtain

$$(4.8)\qquad \begin{aligned}&(\pi t)^{1/2}T(t)x\\ &=\sum_{j=0}^{k}\binom{k}{j}b^{k-j}\left(\frac{-1}{2t}\right)^{j}\int_0^\infty H_{2j}(s/\sqrt{2t})\exp\left(-\frac{s^2}{4t}\right)C(s)R(b;A)^k x\,ds.\end{aligned}$$

Now we evaluate the integral on the right-hand side of (4.8):

$$I_j=(\pi|t|)^{-1/2}\left\|\int_0^\infty H_{2j}(s/\sqrt{2t})\exp\left(-\frac{s^2}{4t}\right)C(s)y\,ds\right\|,\quad y=R(b;A)^k x.$$

Let $\tilde{H}_{2j}(r)$ be the polynomial satisfying

$$|H_{2j}(\zeta)|\leqq \tilde{H}_{2j}(|\zeta|),\qquad \zeta\in\mathbf{C}.$$

Then it follows from Theorem 3.4(b) and (4.3) that

$$\begin{aligned}I_j&\leqq M(\pi|t|)^{-1/2}\int_0^\infty \tilde{H}_{2j}(s/\sqrt{2|t|})\exp\left(-\frac{\sin\varepsilon}{4|t|}s^2+\omega s\right)ds\,\|y\|_k\\ &\leqq N\,\|x\|\exp\left(\frac{\omega^2|t|}{\sin\varepsilon}\right)\int_0^\infty \tilde{H}_{2j}(r)\exp\left[-\frac{\sin\varepsilon}{2}\left(r-\frac{\omega\sqrt{2|t|}}{\sin\varepsilon}\right)^2\right]dr\\ &\leqq N\,\|x\|\exp\left(\frac{\omega^2\operatorname{Re}t}{\sin^2\varepsilon}\right)\int_{-\infty}^\infty \tilde{H}_{2j}\left(s+\frac{\omega\sqrt{2|t|}}{\sin\varepsilon}\right)\exp\left(-\frac{\sin\varepsilon}{2}s^2\right)ds,\end{aligned}$$

where $N=(2/\pi)^{1/2}M\cdot\Sigma_{j=0}^{k}\|A^jR(b;A)^k\|$ and

$$p_j(|t|):=\int_{-\infty}^\infty \tilde{H}_{2j}\left(s+\frac{\omega\sqrt{2|t|}}{\sin\varepsilon}\right)\exp\left(-\frac{\sin\varepsilon}{2}s^2\right)ds$$

is a polynomial of degree j. Thus we obtain (4.6).

Next let $x\in D(A^{k+1})$ and $b\in\rho(A)$ $(b>\omega^2)$. Then it follows from (4.5) and (4.4) that

$$(\pi t)^{1/2}\frac{d}{dt}T(t)x=\frac{1}{2t}\int_0^\infty \frac{d^2}{dr^2}\left[\exp\left(-\frac{r^2}{2}\right)\right]\Big|_{r=s/\sqrt{2t}}C(s)x\,ds.$$

We see from the calculation in the proof of (4.6) that

$$\begin{aligned}&(\pi t)^{1/2}(d/dt)T(t)x\\ &=\frac{1}{2t}\int_0^\infty\left(b-\frac{1}{2t}\frac{d^2}{dr^2}\right)^k\frac{d^2}{dr^2}\left[\exp\left(-\frac{r^2}{2}\right)\right]\Big|_{r=s/\sqrt{2t}}C(s)y\,ds\end{aligned}$$

$$= \int_0^\infty \sum_{j=0}^{k} \binom{k}{j} b^{k-j} \frac{(-1)^j}{(2t)^{j+1}} \left(\frac{d^2}{dr^2}\right)^{j+1} \left[\exp\left(-\frac{r^2}{2}\right)\right]\Bigg|_{r=s/\sqrt{2t}} C(s)yds$$

$$= \sum_{j=0}^{k} \binom{k}{j} b^{k-j} \frac{(-1)^j}{(2t)^{j+1}} \int_0^\infty H_{2(j+1)}(s/\sqrt{2t}) \exp\left(-\frac{s^2}{4t}\right) C(s)yds,$$

where $y = R(b;A)^k x$. Evaluating the integral I_{j+1}, we obtain

$$\left\| \frac{d}{dt} T(t)x \right\| \leqq N \| x \| \exp\left(\frac{\omega^2 \operatorname{Re} t}{\sin^2 e}\right) \sum_{j=0}^{k} \binom{k}{j} b^{k-j} (2t)^{-(j+1)} p_{j+1}(|t|). \quad \text{Q.E.D.}$$

$T(t)x$ was first defined for $x \in D(A^k)$ by the abstract Weierstrass formula (4.1). Now, by virtue of (4.8), we can define $T(t)x$ for an arbitrary $x \in X$:

$$T(t)x = \sum_{j=1}^{k} a_j^k(t) \int_0^\infty H_{2j}(s/\sqrt{2t}) \exp\left(-\frac{s^2}{4t}\right) C(s)R(b;A)^k xds,$$

where $a_j^k(t) = (\pi t)^{-1/2} \binom{k}{j} b^{k-j} (-1/2t)^j$. Both definitions are equivalent for $x \in D(A^k)$. Also we see from (4.6) that $T(t) \in B(X)$ for $\operatorname{Re} t > 0$.

Proposition 4.5. *Let $\{T(t); |\arg t| < \pi/2\}$ be the family in $B(X)$ defined as above. Then $\{T(t)\}$ forms a holomorphic semigroup on X having the properties* (a)–(d) *in Lemma* 4.1 *and*

(e) $AT(t) \in B(X)$, $\operatorname{Re} t > 0$,

(f) $(d/dt)T(t) = AT(t)$, $\operatorname{Re} t > 0$,

(g) $A^j T(t)x = T(t)A^j x$, $x \in D(A^j)$, $\operatorname{Re} t > 0$.

Proof. By definition $T(t)$ is holomorphic in the right half plane (in the sense of norm). The semigroup property of $\{T(t)\}$ follows from Lemma 4.3(c). Therefore $\{T(t)\}$ forms a holomorphic semigroup on X.

(e) It follows from Lemma 4.3(a) and (4.7) that $AT(t)$ is densely defined and bounded. Noting that $AT(t)$ is closed, we see that $AT(t) \in B(X)$.

(f) It follows from the holomorphy of $\{T(t)\}$ that $(d/dt)T(t) \in B(X)$ for $\operatorname{Re} t > 0$. In view of (e) and Lemma 4.3(a) we obtain (f).

(g) It suffices to show the case $j = 1$. Let $x \in D(A)$. Then, since $D(A^{k+1})$ is a core for A (see Lemma 2.3), we can find a sequence $\{x_n\}$ in $D(A^{k+1})$ such that $x_n \to x$ and $Ax_n \to Ax$ $(n \to \infty)$. By Lemma 4.3(b) we have

$$AT(t)x_n = T(t)Ax_n, \qquad \operatorname{Re} t > 0.$$

Going to the limit $n \to \infty$, we obtain the desired equality. Q.E.D.

DEFINITION 4.6. The holomorphic semigroup $\{T(t)\}$ obtained by Proposition 4.5 is called an *abstract Weierstrass semigroup* on X associated with a family $\{C(t)\}$ of generalized solution operators for ACP.

Let $\{T(t); |\arg t| < \pi/2\}$ be an abstract Weierstrass semigroup on X. In the rest of this section we shall show that for any ε $(0<\varepsilon<\pi/2)$, $\{T(t); |\arg t|<(\pi/2)-\varepsilon\}$ forms a semigroup of class (H_k) introduced in Section 2.

Let us introduce the useful notations:

$$X_0 := \bigcup_{t>0} T(t)[X], \quad \Sigma := \{x \in X;\ \| T(t)x - x \| \to 0\ (t \to +0)\}.$$

Then we see from Lemma 4.1(c) that

$$D(A^k) \subset \Sigma. \tag{4.9}$$

LEMMA 4.7. *Let $\{T(t); |\arg t|<\pi/2\}$ be an abstract Weierstrass semigroup on X. Then $\{T(t); t>0\}$ satisfies the conditions* (i) *and* (ii) *in Definition* 2.6.

PROOF. (i) Suppose that $T(t)x = 0$ for all $t > 0$. Let $b \in \rho(A)$. Then it follows from (4.9) and Proposition 4.5(g) that

$$R(b;A)^k x = \lim_{t\to+0} T(t)R(b;A)^k x = \lim_{t\to+0} R(b;A)^k T(t)x = 0.$$

Therefore we obtain $x = 0$.

(ii) Since X_0 is dense in Σ, (4.9) implies that X_0 is dense in X. Q.E.D.

Now let m be a nonnegative integer and $\lambda \in \mathbf{C}$ with $\operatorname{Re} \lambda > \omega^2$. Here ω is the constant in condition $(\mathrm{I}_{\mathrm{real}})$. We define

$$R_m(\lambda)x = \frac{1}{m!} \int_0^\infty t^m e^{-\lambda t} T(t)x\,dt \tag{4.10}$$

whenever the limit exists. Then it follows from (4.6) that for every $x \in X_0$, $R_m(\lambda)x$ makes sense (the factor $\sin^2 \varepsilon$ in (4.6) can be replaced by one if t is real; cf. Lemma 4.1(a)).

LEMMA 4.8. *Let $R_m(\lambda)$ be as above. Then*

$$R_m(\lambda)x = R(\lambda;A)^{m+1}x, \qquad x \in X_0,\ \ \lambda > \omega^2. \tag{4.11}$$

PROOF. First we show that

$$(4.12) \qquad R_0(\lambda)x = R(\lambda; A)x, \qquad x \in X_0, \quad \lambda > \omega^2.$$

Let $x \in X_0$. Then there is $h > 0$ such that $x = T(h)y$, $y \in X$. It follows from Proposition 4.5(f) and (g) that

$$(d/dt)[e^{-\lambda t}T(t+h)R(b; A)y] = -e^{-\lambda t}T(t+h)y, \qquad t > 0,$$

and hence

$$\begin{aligned} R(\lambda; A)T(h)y &= T(h)R(\lambda; A)y \\ &= \int_0^\infty e^{-\lambda t}T(t)T(h)y\,dt = R_0(\lambda)T(h)y, \qquad \lambda > \omega^2. \end{aligned}$$

Therefore we obtain (4.12).

If $x \in X_0$ and $\lambda > \omega^2$, then we have

$$R_m(\lambda)x = \frac{(-1)^m}{m!}\left(\frac{d}{d\lambda}\right)^m R_0(\lambda)x.$$

Hence we obtain (4.11). Q.E.D.

LEMMA 4.9. *Let $\{T(t); |\arg t| < \pi/2\}$ be as in Lemma 4.7. Let $b > \omega^2$. Then*

(iii′) *For any ε $(0 < \varepsilon \leqq \pi/2)$ there is $M_\varepsilon > 0$ such that*

$$(4.13) \quad \| T(t)R_{k-1}(b)x \| \leqq M_\varepsilon \exp\left(\frac{\omega^2}{\sin^2\varepsilon}\operatorname{Re} t\right) \| x \|, \ x \in X_0, \ |\arg t| \leqq \frac{\pi}{2} - \varepsilon,$$

where

$$M_\varepsilon = \frac{2M}{\sqrt{\sin \varepsilon}} \sum_{j=0}^{k} \| A^j R(b; A)^k \|.$$

(iv) *There is $K > 0$ such that*

$$\| R_0(b)x \| \leqq K \| x \|, \qquad x \in X_0.$$

PROOF. (iii′) We see from (4.1) and (4.11) with $\lambda = b$ and $m = k-1$ that

$$T(t)R_{k-1}(b)x = \frac{1}{\sqrt{\pi t}}\int_0^\infty \exp\left(-\frac{s^2}{4t}\right) C(s)R(b; A)^k x\,ds, \qquad x \in X_0.$$

Therefore (4.13) follows from Lemma 4.1(b).

(iv) is a consequence of (4.12) with $\lambda = b$. Q.E.D.

REMARK 4.10. Unfortunately we could not replace $\omega^2/\sin^2 \varepsilon$ in (4.13) by just ω^2.

PROPOSITION 4.11. *Let $\{T(t); |\arg t| < \pi/2\}$ be an abstract Weierstrass semigroup on X. Then for any ε $(0 < \varepsilon < \pi/2)$, $\{T(t); |\arg t| < (\pi/2) - \varepsilon\}$ forms a semigroup of class (H_k) with complete infinitesimal generator A.*

PROOF. We see from Lemma 4.7 and 4.9 that $\{T(t); |\arg t| < (\pi/2) - \varepsilon\}$ forms a semigroup of class (H_k) (the constant ω in Definition 2.6 depends on $\varepsilon : \omega(\varepsilon) = \omega^2/\sin^2 \varepsilon$). So it remains to show that A is the complete infinitesimal generator of $\{T(t)\}$.

Let A_0 be the infinitesimal generator of $\{T(t); t > 0\}$. Then condition (i) implies that A_0 has the closure $\overline{A}_0$ and

$$R_0(\lambda)x = (\lambda - \overline{A}_0)^{-1}x, \qquad x \in X_0, \quad \lambda > \omega^2$$

(see Lemma A.4 below). It follows from (4.12) that

$$(\lambda - \overline{A}_0)^{-1}x = R(\lambda; A), \qquad x \in X_0, \quad \lambda > \omega^2.$$

Since X_0 is dense in X, we see that $A = \overline{A}_0$. Q.E.D.

Finally we state a necessary condition for ACP on $D(A^{k+1})$ to be exponentially wellposed.

THEOREM 4.12. *Let A be a closed and densely defined linear operator in X satisfying*

$(\mathrm{I}_{\text{real}})$ *there is a constant $\omega > 0$ such that*

$$\rho(A) \supset \{\lambda^2 \in \mathbf{R}; \lambda > \omega\},$$

$(\mathrm{II}_{\text{real}})$ *there is a constant $M > 0$ such that for $x \in D(A^k)$,*

$$\| (d/d\lambda)^{n-1}[\lambda R(\lambda^2; A)x] \| \leqq M(n-1)!(\lambda - \omega)^{-n} \| x \|_k, \quad \lambda > \omega, \ n \in \mathbf{N}.$$

Then

(a) *$\rho(A)$ contains the sector $|\arg(\zeta - \omega^2)| < \pi$,*

(b) *for any ε $(0 < \varepsilon \leqq \pi/2)$ there is a constant $M'_\varepsilon > 0$ such that for $x \in D(A^k)$,*

$$\| (\zeta - A)^{-1}x \| \leqq M'_\varepsilon \left| \zeta - \frac{\omega^2}{\sin^2 \varepsilon} \right|^{-1} \| x \|_k, \quad \left| \arg\left(\zeta - \frac{\omega^2}{\sin^2 \varepsilon}\right) \right| < \pi - \varepsilon.$$

PROOF. Under the assumption ACP on $D(A^{k+1})$ is exponentially wellposed (see Theorem 3.4). Consequently, $\rho(A)$ contains the right half-plane $\operatorname{Re} \lambda > \omega^2$. Moreover, we can construct an abstract Weierstrass semigroup on X. So we see from Proposition 4.11 that A is the complete infinitesimal generator of an (H_k)-semigroup $\{T(t); |\arg t| < (\pi/2) - \varepsilon\}$ for any ε $(0 < \varepsilon < \pi/2)$. Therefore it follows from Theorem A.2(b$_1$) below that $\rho(A)$ contains

$$\bigcup_{0<\varepsilon<\pi/2} \left\{\zeta \in \mathbf{C};\ \left|\arg\left(\zeta - \frac{\omega^2}{\sin^2 \varepsilon}\right)\right| < \pi - \varepsilon\right\}.$$

Hence we obtain (a).

(b) is a consequence of Theorem A.2(b$_2$). Q.E.D.

REMARK 4.13. Recently Watanabe [17] found fairly weak conditions under which ACP has a unique solution for initial values in a dense subset Y of X. The set Y becomes a linear topological space identified as an abstract Gevrey space associated with A. Conditions in this paper are really stronger than those in [17]; however, as its consequence we could obtain a clear continuous dependence of solutions on initial data.

Appendix

Let us begin with the definition of a semigroup of class (H_n) which was introduced in [11].

DEFINITION A.1. Let $n \in \mathbf{N}$. Then a holomorphic semigroup $\{T(t); |\arg t| < \alpha\}$ $(0 < \alpha \leqq \pi/2)$ on X is said to be of class (H_n) if it satisfies the following conditions:

(α_1) $T(t)x = 0$ for all $t > 0$ implies that $x = 0$.

(α_2) For each $\varepsilon > 0$ there is a constant $C_\varepsilon > 0$ such that

$$\| t^n T(t) \| \leqq C_\varepsilon e^{\omega \operatorname{Re} t} \qquad \text{for } |\arg t| \leqq \alpha - \varepsilon,$$

where ω is a real constant.

(α_3) $X_0 := \bigcup_{t>0} T(t)[X]$ is dense in X.

(α_4) There is $b > \omega$ such that $R(R_{n+1}(b)) \subset R(R_n(b))$, where, as in Section 2,

$$R_m(\lambda)x = \frac{1}{m!}\int_0^\infty t^m e^{-\lambda t} T(t)x\,dt, \qquad m \in \mathbf{N} \cup \{0\},$$

whenever the limit exists, and $R(R_n(b))$ is the range of $R_n(b)$.

(α_5) For each $\varepsilon > 0$ there is a constant $K_\varepsilon > 0$ such that $\| T(t)R_{n-1}(b)x \| \leqq K_\varepsilon \| x \|$ for $x \in X_0$ and $0 < |t| < 1$ with $|\arg t| \leqq \alpha - \varepsilon$.

A characterization for the complete infinitesimal generator of a semigroup of class (H_n) is given by

THEOREM A.2 ([11], Theorem 1.6). *Let A be a closed linear operator with domain $D(A)$ and range $R(A)$ in X and let $n \in \mathbf{N}$. Then A is the complete infinitesimal generator of a semigroup of class (H_n) if and only if the following three conditions are satisfied*:

(b_1) *There is a constant $\omega \in \mathbf{R}$ such that $\rho(A)$ contains the sector $|\arg(\zeta - \omega)| < (\pi/2) + \alpha$ for some $0 < \alpha \leqq \pi/2$.*

(b_2) *For each $\varepsilon > 0$ there is a constant $M_\varepsilon > 0$ such that for $x \in D(A^n)$,*

$$\| R(\zeta; A)x \| \leqq M_\varepsilon |\zeta - \omega|^{-1} \| x \|_n, \quad |\arg(\zeta - \omega)| \leqq (\pi/2) + \alpha - \varepsilon.$$

(b_3) *$D(A)$ is dense in X.*

The purpose of this appendix is to give a new characterization (equivalent definition) for semigroups of class (H_n):

THEOREM A.3 (see Definition 2.6). *A holomorphic semigroup $\{T(t); |\arg t| < \alpha\}$ is of class (H_n) if and only if it satisfies the following conditions*:

(i) *$T(t)x = 0$ for all $t > 0$ implies that $x = 0$.*

(ii) *$X_0 := \bigcup_{t>0} T(t)[X]$ is dense in X.*

(iii) *For each ε $(0 < \varepsilon \leqq \alpha)$ there is $L_\varepsilon > 0$ such that*

$$\| T(t)R_{n-1}(b)x \| \leqq L_\varepsilon e^{\omega \operatorname{Re} t} \| x \|, \qquad x \in X_0, \quad |\arg t| \leqq \alpha - \varepsilon,$$

where b and ω are constants satisfying

$$b > \omega > \omega_0 := \lim_{t \to \infty} t^{-1} \log \| T(t) \| .$$

(iv) *There is $K > 0$ such that $\| R_0(b)x \| \leqq K \| x \|$ for $x \in X_0$.*

To prove Theorem A.3 we need some preparations. We use the following notations:

$$X_0 := \bigcup_{t>0} T(t)[X], \qquad \omega_0 := \lim_{t \to \infty} t^{-1} \log \| T(t) \| ,$$

$$\Sigma := \{x \in X;\ \| T(t)x - x \| \to 0 \text{ as } t \to +0\}.$$

By definition X_0 is dense in Σ, and it is well known that ω_0 is finite or $-\infty$.

Here we state two lemmas which were shown in [11].

LEMMA A.4 ([11], Lemmas 2.1 and 3.2). *Let $\{T(t); t>0\}$ be a semigroup on X satisfying condition* (α_1). *Then the infinitesimal generator A_0 is closable and for $x \in \Sigma$ and $\lambda > \omega_0$*

$$(\lambda - A)^{-m}x = R_{m-1}(\lambda)x, \qquad m \in \mathbf{N}, \tag{A.1}$$

where $A = \overline{A_0}$ is the complete infinitesimal generator of $\{T(t)\}$.

LEMMA A.5 ([11], Lemmas 3.4 and 3.5). *Let A be the complete infinitesimal generator of a semigroup of class* (H_n). *Then $b \in \rho(A)$ and $D(A^n) \subset \Sigma$.*

Now we prove the "only if" part of Theorem A.3. Since conditions (i) and (ii) are the same as conditions (α_1) and (α_3), it suffices to show that conditions (iii) and (iv) follows from Definition A.1:

LEMMA A.6. *Let $\{T(t); |\arg t| < \alpha\}$ be a semigroup of class* (H_n) *in the sense of Definition* A.1. *Then it satisfies conditions* (iii) *and* (iv).

PROOF. Let A be the complete infinitesimal generator of $\{T(t)\}$ (the existence of which is guaranteed by Lemma A.4). Then we see from (A.1) and Lemma A.5 that $b \in \rho(A)$ and

$$R_0(b)x = (b-A)^{-1}x = R(b;A)x, \qquad x \in X_0.$$

So we obtain condition (iv) with $K = \| R(b;A) \|$.

Next we derive condition (iii). Let $x \in X_0$ and $|t| \geqq 1$ with $|\arg t| \leqq \alpha - \varepsilon$. Then we shall show that

$$\| T(t)R_{n-1}(b)x \| \leqq \frac{C_\varepsilon}{(n-1)!(b-\omega)} e^{\omega \operatorname{Re} t} \| x \|, \tag{A.2}$$

where C_ε is the constant in condition (α_2). By the definition of $R_{n-1}(b)$ we have

$$\begin{aligned} &(n-1)!\, T(t)R_{n-1}(b)x \\ &= \int_0^\infty e^{-bs}s^{n-1}T(s+t)x\,ds \\ &= \int_0^\infty e^{-bs}(t+s)^{-1}[s/(t+s)]^{n-1}(s+t)^nT(s+t)x\,ds, \quad x \in X_0. \end{aligned}$$

Since $|t+s| \geqq |t| \geqq 1$ and $|t+s| \geqq s$ for $s \geqq 0$ and $|t| \geqq 1$ with $|\arg t| \leqq \alpha - \varepsilon$, (A.2) follows from condition (α_2). In view of condition (α_5) we obtain condition (iii). Q.E.D.

In the rest of this appendix we shall prove the "if" part of Theorem A.3.

LEMMA A.7. *Let* $\{T(t); |\arg t| < \alpha\}$ *be a holomorphic semigroup on* X *satisfying conditions* (i)–(iv) *in Theorem* A.3. *Let* A *be the complete infinitesimal generator of* $\{T(t)\}$ *(see Lemma* A.4). *Then* $b \in \rho(A)$ *and*

$$\| T(t)R(b;A)^n \| \leqq L_\varepsilon e^{\omega \operatorname{Re} t}, \quad |\arg t| \leqq \alpha - \varepsilon, \tag{A.3}$$

where $R(b;A) = (b-A)^{-1} \in B(X)$, *and for* $x \in D(A^n)$

$$\| T(t)x - x \| \to 0 \qquad \text{as } t \to 0 \quad \text{with } |\arg t| \leqq \alpha - \varepsilon. \tag{A.4}$$

PROOF. We see from (A.1) and condition (iv) that

$$\| (b-A)^{-1}x \| \leqq K \| x \| \qquad \text{for } x \in X_0.$$

Since $(b-A)^{-1}$ is closed, it follows from condition (ii) that $(b-A)^{-1} \in B(X)$, i.e., $b \in \rho(A)$ and

$$R(b;A)x = R_0(b)x \qquad \text{for } x \in X_0. \tag{A.5}$$

So (A.3) follows from (A.1) and condition (ii) and (iii). Furthermore, we have

$$T(t)R(b;A) = R(b;A)T(t), \qquad |\arg t| < \alpha.$$

In fact, $T(t)R_0(b)x = R_0(b)T(t)x$ for $x \in X_0$. Therefore, it follows from (A.5) that for $x \in X_0$ and $|\arg t| < \alpha$,

$$\begin{aligned} T(t)R(b;A)^{n+1}x &= R_0(b)T(t)R(b;A)^n x \\ &= \int_0^\infty e^{-bs}T(s+t)R(b;A)^n x\,ds; \end{aligned} \tag{A.6}$$

note that $T(t)R(b;A)^n x \in X_0$ for $x \in X_0$.

Now let $|t| \leqq 1$ with $|\arg t| \leqq \alpha - \varepsilon$. Then $\operatorname{Re} t \leqq 1$ and we see from (A.3) that

$$\| e^{-bs}T(s+t)R(b;A)^n x \| \leqq e^{\omega} L_\varepsilon e^{-(b-\omega)s} \| x \|, \qquad x \in X_0, \quad s > 0.$$

Therefore, it follows from (A.6) and condition (ii) that for any $x \in X$,

$$T(t)R(b;A)^{n+1}x = \int_0^\infty e^{-bs}T(s+t)R(b;A)^n x\,ds. \tag{A.7}$$

Applying the dominated convergence theorem, we see that

$$\begin{aligned} T(t)R(b;A)^{n+1}x \to \int_0^\infty e^{-bs}T(s)R(b;A)^n x\,ds \\ = R_0(b)R(b;A)^n x = R(b;A)^{n+1}x \end{aligned}$$

as $t \to 0$ with $|\arg t| \leqq \alpha - \varepsilon$. Consequently, we obtain (A.4) with $x \in D(A^{n+1})$, i.e.,

$$\| T(t)R(b;A)^n x - R(b;A)^n x \| \to 0 \qquad \text{for } x \in D(A)$$

as $t \to 0$ with $|\arg t| \leqq \alpha - \varepsilon$. Thus, (A.4) follows from (A.3); note that $D(A)$ is dense in X by condition (ii) (see Hille–Phillips [7], Theorem 10.3.1). Q.E.D.

LEMMA A.8. *Let $\{T(t)\}$ and A be as in Lemma* A.7. *Then for $\zeta \in \mathbf{C}$ with* $|\arg(\zeta - \omega)| < (\pi/2) + \alpha$,

$$\text{(A.8)} \qquad (\zeta - A)^{-1} \textit{ exists} \quad \textit{and} \quad R(\zeta - A) \supset D(A^n).$$

PROOF. Let $\lambda \in \mathbf{C}$ with $\operatorname{Re}\lambda > 0$ and let $\theta \in \mathbf{R}$ be fixed, with $|\theta| < \alpha$. Setting $\zeta = \omega + \lambda e^{-i\theta}$ and noting that $b \in \rho(A)$ (see Lemma A.7), we have for $x \in D(A_0)$

$$(d/dt)[e^{-\zeta t}T(t)R(b;A)^n x] = -e^{-\zeta t}T(t)R(b;A)^n(\zeta - A_0)x.$$

Integrating this equality along the ray $t = re^{i\theta}$ from $r = s > 0$ to $r = \infty$, we obtain for $x \in D(A)$

$$\begin{aligned} &\exp(-\lambda s - \omega s e^{i\theta})T(se^{i\theta})R(b;A)^n x \\ &\quad = e^{i\theta}\int_s^\infty \exp(-\lambda r - \omega r e^{i\theta})T(re^{i\theta})R(b;A)^n(\zeta - A)x\,dr. \end{aligned}$$

In fact, we can replace A_0 by its closure A because of (A.3). Going to the limit $s \to +0$, we see from (A.4) that

$$\text{(A.9)} \quad e^{-i\theta}R(b;A)^n x = \int_0^\infty \exp(-\lambda r - \omega r e^{i\theta})T(re^{i\theta})R(b;A)^n(\zeta - A)x\,dr.$$

This shows that $(\zeta - A)^{-1}$ exists for $|\arg(\zeta - \omega)| = |\arg \lambda e^{-i\theta}| < (\pi/2) + \alpha$.

Now let $y \in X$ and set $x = R(b;A)y$ in (A.9). Then, since $(\zeta - A)R(b;A) \in B(X)$, we see from (A.9) that

$$\text{(A.10)} \quad e^{-i\theta}R(b;A)^n x = (\zeta - A)\int_0^\infty \exp(-\lambda r - \omega r e^{i\theta})T(re^{i\theta})R(b;A)^n x\,dr.$$

Since $D(A)$ is dense in X, it follows from the closedness of A that (A.10) holds for all $x \in X$. Thus, we obtain (A.8). Q.E.D.

Now the proof of the "if" part of Theorem A.3 is completed by

LEMMA A.9. *Let $\{T(t); |\arg t| < \alpha\}$ be a holomorphic semigroup on X satisfying conditions* (i)–(iv) *in Theorem* A.3. *Then $\{T(t)\}$ is of class* (H_n).

PROOF. Let A be the complete infinitesimal generator of $\{T(t)\}$ satisfying conditions (i)–(iv). To prove the assertion we will go rather indirectly. Namely, let us first show that A satisfies conditions (b_1)–(b_3) in Theorem A.2.

As noted at the end of the proof of Lemma A.7, $D(A)$ is dense in X. This is condition (b_3). In view of Lemma A.10 below, condition (b_1) is a consequence of Lemmas A.7 and A.8. Therefore, it follows from (A.10) that for $x \in X$,

$$R(\zeta;A)R(b;A)^n x = e^{i\theta}\int_0^\infty \exp(-\lambda r - \omega re^{i\theta})T(re^{i\theta})R(b;A)^n x dr.$$

So, we see from (A.3) that for $\operatorname{Re}\lambda > 0$ and $|\theta| \leqq \alpha - \varepsilon/2$,

$$\| R(\zeta;A)R(b;A)^n x \| \leqq L_{\varepsilon/2}(\operatorname{Re}\lambda)^{-1} \| x \| .$$

Here $\operatorname{Re}\lambda \geqq |\lambda| \sin(\varepsilon/2)$ if we take $|\arg\lambda| \leqq (\pi/2) - \varepsilon/2$. It then follows that for $|\arg\lambda| \leqq (\pi/2) - \varepsilon/2$ and $|\theta| \leqq \alpha - \varepsilon/2$,

$$\| R(\zeta;A)R(b;A)^n x \| \leqq \frac{L_{\varepsilon/2}}{\sin(\varepsilon/2)} |\lambda|^{-1} \| x \| , \qquad x \in X.$$

Since $\zeta - \omega = \lambda e^{-i\theta}$, we have $|\arg(\zeta - \omega)| \leqq (\pi/2) + \alpha - \varepsilon$. Thus, we obtain condition (b_2).

Consequently, A is the complete infinitesimal generator of a semigroup $\{\tilde{T}(t); |\arg t| < \alpha\}$ of class (H_n); as shown above $\{T(t)\}$ and $\{\tilde{T}(t)\}$ have the same semi-angle α. But, it follows from (A.4) and Lemma A.5 that $D(A^n)$ is contained in $\Sigma \cap \tilde{\Sigma}$, where $\tilde{\Sigma}$ is the continuity set of $\{\tilde{T}(t)\}$. Therefore, we see from (A.1) and condition (b_1) that for $x \in D(A^n)$,

$$\int_0^\infty e^{-\lambda t}T(t)x dt = R_0(\lambda)x = R(\lambda;A)x$$

$$= \tilde{R}_0(\lambda)x := \int_0^\infty e^{-\lambda t}\tilde{T}(t)x dt, \qquad \lambda > \omega > \omega_0.$$

This implies that $T(t)x = \tilde{T}(t)x$ for $x \in D(A^n)$. Since $D(A^n)$ is dense in X, we can obtain the desired conclusion. Q.E.D.

The final lemma is the one quoted above.

LEMMA A.10 ([11], Lemma 2.4). *Let A be a closed linear operator in X and let $n \in \mathbf{N}$. Suppose that there is $\omega \in \mathbf{R}$ such that $(\zeta - A)^{-1}$ exists and $R(\zeta - A) \supset D(A^n)$ for $|\arg(\zeta - \omega)| < (\pi/2) + \beta$, where β is a constant with $0 \leqq \beta \leqq \pi/2$. If $\rho(A) \neq \varnothing$ then $\rho(A)$ contains the sector $|\arg(\zeta - \omega)| < (\pi/2) + \beta$.*

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